Towards a general theory for coupling functions allowing persistent synchronization

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Abstract. We study coupling functions that allow for persistent synchronization in connected complex networks and any isolated system dynamics that possesses global solutions and bounded Jacobian evaluated along such solutions. We prove that the set of coupling functions leading to stable synchronization is open and that any coupling function whose linear part has eigenvalues with positive real part leads to system towards synchronization.

1. Introduction

Network synchronization is observed to occur in a broad range of applications in physics [1], neuroscience [2, 3, 4, 5], ecology [6], and life sciences [7]. During the last fifty years, empirical studies of real complex systems have led to a deep understanding of the structure of networks, the isolated dynamics of individual elements [8], the interaction structure [9, 10], and the interaction properties between oscillators, that is, the coupling function [11, 12].

The stability of network synchronization is a balance between the isolated dynamics and the coupling function. Past research suggests that in networks of identical oscillators with interaction akin to diffusion, under mild conditions on the isolated dynamics, the coupling function dictates the synchronization properties of the network [13, 14, 15]. However, it still remains an open problem to describe the class of coupling functions that lead the network to global synchronization for any isolated dynamics satisfying mild conditions.

Our work contributes to the development a general theory for coupling functions that allow for persistent synchronization for a connected complex network. The coupling functions under consideration appear in a variety of synchronization models on networks (such as the Kuramoto models [12] and its extensions [19, 20]).

More precisely, we consider the dynamics of a network of n identical elements with interaction akin to diffusion, described by

$$\dot{x}_i = f(t, x_i) + \alpha \sum_{j=1}^{n} W_{ij} H(x_j - x_i)\,, \tag{1}$$

where α is the overall coupling strength, and the matrix $W = (W_{ij})_{i,j\in\{1,\ldots,n\}}$ describes the interaction structure of the network, i.e. W_{ij} measures the strength of interaction between the nodes i and j. We make the following two assumptions for the function $f\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$, and the coupling function $H\colon \mathbb{R}^m \to \mathbb{R}^m$.

Assumption A1. *The function f is continuous in the first argument and continuously differentiable in the second argument, and there exists an open set $U \subset \mathbb{R}^m$ with C^1 boundary that is ε-inflowing invariant for some $\varepsilon > 0$, i.e. the vector field f is pointing strictly inward at ∂U, uniformly bounded away from zero (see Definition 7). Moreover, the Jacobian $D_x f$ with respect to $x \in \mathbb{R}^m$ is uniformly continuous and bounded on U, i.e. there exists $B > 0$ such that*

$$\|D_x f(t, x)\| \le B \quad \text{for all } t \in \mathbb{R} \text{ and } x \in U\,.$$

Note that if $\bar{U}$ is compact, then uniformity of the inflowing invariance condition as well as the uniform continuity of $D_x f$ and existence of a bound B follow automatically.

Assumption A2. *The coupling function H is continuously differentiable with $H(0) = 0$. We define $\Gamma := DH(0)$ and denote the (complex) eigenvalues of Γ by β_i, $i \in \{1, \ldots, m\}$.*

The network structure plays a central role for the synchronization properties. We consider the intensity of the i-th node $S_i = \sum_{j=1}^{n} W_{ij}$, and define the positive definite matrix $S := \mathrm{diag}(S_1, \ldots, S_n)$. Then the so-called *Laplacian* reads as

$$L = S - W\,.$$

Let θ_i, $i \in \{1, \ldots, n\}$, denote the eigenvalues of L. Note that $\theta_1 = 0$ is an eigenvalue with eigenvector $(1, 1, \ldots, 1)$. The multiplicity of this eigenvalue equals the number of connected components of the network. The last hypothesis is a crucial sufficient condition for synchronization.

Assumption A3. *We suppose that*

$$\gamma := \min_{2 \le i \le n,\, 1 \le j \le m} \mathrm{Re}(\theta_i \beta_j) > 0\,,$$

where $\mathrm{Re}(z)$ *denotes the real part of a complex number* z.

The dynamics of such a diffusive model can be intricate. Indeed, even if the isolated dynamics possesses a globally stable fixed point, the diffusive coupling can lead to instability of the fixed point and the system can exhibit an oscillatory behavior [21].

Note that due to the diffusive nature of the coupling, if all oscillators start with the same initial condition, then the coupling term vanishes identically. This ensures that the globally synchronized state $x_1(t) = x_2(t) = \ldots = x_n(t) = s(t)$ is an invariant state for all coupling strengths α and all choices of coupling functions H, and we call the set

$$M := \{x \in (\mathbb{R}^m)^n : x_1 = \ldots = x_n\}$$

the *synchronization manifold*. The main result of this paper is a proof that under the general conditions given above and α sufficiently large, the synchronization manifold S is exponentially stable.

Theorem 1 (synchronization)**.** *Consider the network of diffusively coupled equations (1) satisfying A1–A3. Then there exists an* $\alpha_c = \alpha_c(f, \Gamma)$ *such that for all coupling strengths*

$$\alpha > \frac{\alpha_c}{\gamma}\,,$$

the network is locally uniformly synchronized, i.e. there exist $C, \delta > 0$ *such that if* $x_i(s) \in U$ *and* $\|x_i(s) - x_j(s)\| \le \delta$ *for any* $i, j \in \{1, \ldots, n\}$*, then*

$$\|x_i(t) - x_j(t)\| \le C e^{-(\alpha\gamma - \alpha_c)(t-s)} \|x_i(s) - x_j(s)\| \quad \textit{for all } t \ge s\,.$$

Hence, the characteristic relaxation time *towards synchronization is given by* $\frac{1}{\alpha\gamma - \alpha_c}$.

Once the bound holds, the local dynamics will converge to the synchronization state and will be stable under small perturbations of the state. This means that the phenomenon of bubbling [22] and riddling [23] which leads to synchronization loss will not be observed, as opposed to the bounds that arise from the master stability function approach. Note that our approach is constructive and provides precise estimates for α_c. These estimates depend on the properties of the matrices Γ and $Df(s(t))$.

Our second main result shows that this synchronization is persistent under perturbation of the isolated nodes. Thereto, consider a network of non-identical nodes described by

$$\dot{x}_i = f_i(t, x_i) + \alpha \sum_{j=1}^{n} W_{ij} H(x_j - x_i), \tag{2}$$

where $f_i(t, x_i) = f(x_i) + g_i(t, x_i)$. Note that in this case, the synchronization manifold S is no longer invariant. We will show in this paper that under the following additional conditions on the perturbation functions g_i, $i \in \{1, \ldots, n\}$, the synchronization manifold is stable in the sense that orbits starting near the synchronization manifold M remain in a neighborhood of M.

Theorem 2 (persistence)**.** *Consider a network of diffusively coupled equations and a choice of $\alpha > \alpha_c/\gamma$ as in Theorem 1. Let (2) describe a perturbation of the network such that*

$$\|g_i(t, x)\| \leq \varepsilon_g \quad \text{for all } t \in \mathbb{R}\,,\ x \in \mathbb{R}^m \text{ and } i \in \{1, \ldots, n\}\,.$$

Then there exist $\varepsilon_g, \delta > 0$ such that if $\|x_i(s) - x_j(s)\| \leq \delta$ for any $i, j \in \{1, \ldots, n\}$, then the network is locally approximately synchronized in the sense that the differences $\|x_i(t) - x_j(t)\|$ still converge at an exponential rate towards close to zero.

See page 16 for a more precise statement of this theorem.

2. Discussion of the main results

This section is devoted to relate our results to the state of the art, explain the assumptions and the ideas of the proofs.

2.1. State of the art

Recent efforts have focused on the role of the coupling function on the stability of the network synchronization. Pecora and collaborators have performed extensive experiments on various types of isolated dynamics and coupling functions [16] to analyze the stability of synchronization. Pogromsky and Nijmeijer showed that if the linear coupling function is conjugated to a positive definite matrix, that is, symmetric and with positive eigenvalues, then it is always possible to synchronize a connected network [17]. Ashwin and coworkers have introduce coupling functions that allow clusters of synchronization in the phase oscillators [18].

2.2. The assumptions

Our assumptions are rather natural. Without assumption *A1* (existence of solutions) we could not speak of stability of the synchronized state. The second part of assumption *A1* (boundedness of the Jacobian) basically implies that with a finite value of α, choosing the coupling matrix properly we are able damp all the instabilities of the vector field and

obtain a stable synchronization state. Assumption *A2* makes it possible to characterize the stability of synchronization behavior by the linearization of the H. Assumption *A3* guarantees that γ is bounded away from zero. If this hypothesis is dropped, γ may become negative and synchronization may no longer be possible. All these ingredients are used to prove the main result.

2.3. Ideas of the proofs

To prove the result we map the synchronization problem to a corresponding fixed point problem. First, we start analyzing diagonalizable Laplacians. It is easy to show that if the coupling function is diagonalizable, then the diagonal dominance, see Proposition 5, implies that the synchronized state corresponds to a uniformly asymptotically stable fixed point. To obtain the claim for general coupling functions we make use of the roughness property associated with the equilibrium point, see Proposition 4. The main aspect here is to approximate the coupling function by a diagonalizable one while keeping control of the contraction rates. To conclude the proof for general Laplacians we prove that the set of diagonalizable Laplacians is dense in the space of Laplacians. From these results and the roughness property the main claim follows.

2.4. The Assumption A3

As far as we are aware condition *A3* has not been put forward in the literature. Therefore, we would like to rephrase this condition when some further requirements are satisfied.

The Spectrum of Γ is positive: If Γ has a spectrum consisting of only real eigenvalues then the condition *A3* has a representation in terms of the Laplacian. In this case, the condition becomes

$$\mathrm{Re}(\theta_i) > 0$$

for every $i \neq 1$, as the Laplacian always has a zero eigenvalue. If the network is connected this eigenvalue is simple and in virtue of the disk theorem a sufficient condition for all eigenvalues to have positive real part is:

The interaction strengths are positive, i.e., $W_{ij} > 0$ if i is connected to j, and zero otherwise.

The Laplacian is Symmetric: This is the most studied case in the literature. The spectrum of the Laplacian is real. Assuming the network is connected we can order the eigenvalues as

$$0 = \theta_1 < \theta_2 \leq \theta_3 \cdots \leq \theta_n$$

In this case the condition *A2* reduces to requiring the real part of the spectrum of Γ to be positive. Moreover, in this case the Laplacian L is diagonalizable by an orthogonal,

similarity transformation, which implies that the bounds on the persistence results reduces to

$$\|x_i(t) - x_j(t)\| \leq \tilde{C}\kappa(Q)\frac{\delta}{\alpha\gamma - \alpha_c}.$$

as $\kappa_2(P) = 1$ and $\tilde{C}$ is a constant depending only on the dimension of the isolated systems.

3. Illustrations

Before proving the main result, we consider two illustrative cases. First, two coupled nonautonomous linear equations. Then, we explore a case of coupled Lorenz systems.

3.1. Nonautonomous Linear Equations

Consider the nonautonomous linear equation

$$\frac{dx}{dt} = A(t)x$$

where

$$A(t) = \begin{pmatrix} -1 - 9\cos^2(6t) + 12\sin(6t)\cos(6t) & 12\cos^2(6t) + 9\sin(6t)\cos(6t) \\ -12\sin^2(6t) + 9\sin(t)\cos(6t) & -1 - 9\sin^2(6t) - 12\sin(6t)\cos(6t) \end{pmatrix}$$

This is a classic example where the eigenvalues do not characterize the stability of the trivial solution of a nonautonomous equation. Indeed, the eigenvalues of $A(t)$ are -1 and -10, even independent of time, however, a direct computation shows that

$$x(t) = \begin{pmatrix} e^{2t}(\cos(6t) + 2\sin(6t)) + 2e^{-13t}(2\cos(6t) - \sin(6t)) \\ e^{2t}(\cos(6t) - 2\sin(6t)) + 2e^{-13t}(2\cos(6t) - \sin(6t)) \end{pmatrix}$$

is an unstable solution of the system. Hence, this system satisfies all the hypotheses. Consider now two diffusively coupled systems

$$\frac{dx_1}{dt} = A(t)x_1 + \alpha\Gamma(x_2 - x_1) \tag{3}$$

$$\frac{dx_2}{dt} = A(t)x_2 + \alpha\Gamma(x_1 - x_2) \tag{4}$$

where Γ is a matrix. According to our main result, it is possible to synchronize these two systems for any coupling matrix with $\beta(\Gamma) > 0$. Consider now the following coupling function

$$\Gamma = \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix},$$

Such a coupling function is in its Jordan form, and is non-diagonalizable. It is possible to synchronize these two systems by setting α properly. Consider the variable $\xi = x_1 - x_2$, its evolution equation reads as

$$\frac{d\xi}{dt} = [A(t) - 2\alpha\Gamma]\xi. \tag{5}$$

Our main result shows that the trivial solution of Eq. (5) is stable if α is large enough.

We integrated Eq. (5) using a sixth order Runge-Kutta method with integration step 0.001. We computed the critical coupling α_c as a function of β, the result can be observed in Fig. 1. The behavior of α_c as a function of β appears to be intricate. For large β we obtain that α_c tends to a constant, however, as we decrease β various changes of behavior can be observed. Although, the problem is linear, the critical

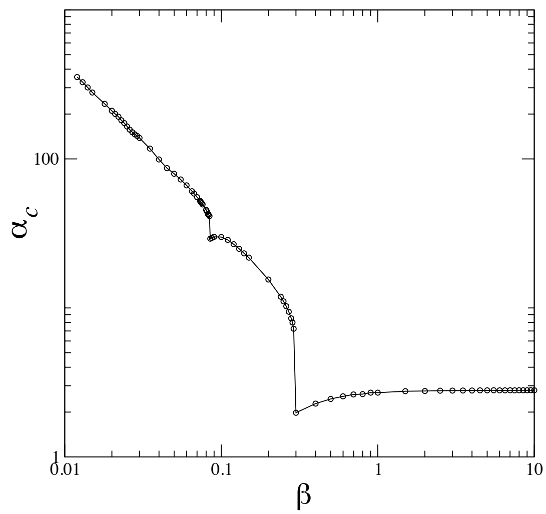


Figure 1. We depict the critical coupling strength α as a function of λ in a $\log - \log$ scale.

coupling strength depends nonlinearly on the parameter β.

3.2. Lorenz Oscillators

Using the notation $x = (u, v, w)^*$, the Lorentz vector field reads

$$f(x) = \begin{pmatrix} \sigma(v-u) \\ u(r-w)-v \\ -bw+uv \end{pmatrix}$$

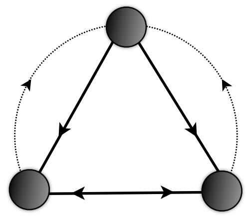
where we choose the classical parameter values $\sigma = 10, r = 28, b = 8/3$. The trajectories of the Lorenz eventually enter a compact set. Therefore, all trajectories of the system exist globally forward in time. Moreover, they accumulate in a neighborhood of a chaotic attractor [24].

Consider a network of three coupled Lorenz systems

$$\frac{dx_i}{dt} = f(x_i) + \alpha \sum_{j=1}^{3} W_{ij} H(x_j - x_i), \tag{6}$$

the interaction matrix W is depicted in Fig. 2.

We shall use two distinct nonlinear coupling functions, the first the associated matrix Γ is positive definite, whereas as in the second Γ is a Jordan block. The specific form of the coupling function can be seen in Fig. 3



$$\boldsymbol{W} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & a \\ a & 1 & 0 \end{pmatrix}$$

Figure 2. The network and its weight matrix. The matrix $L = S - W$ is non-diagonalizable for every $a \neq 1$, here, we choose $a = 1/3$.

We integrated Eq. (6) using a sixth order Runge-Kutta method with integration step 10^{-4}, and computed the critical coupling α_c as a function of β, the result can be observed in Fig. 3. The behavior of α_c is essentially different depending on the diagonalization properties of the associated Γ.

Jaap: I think that the result that $\alpha_c \propto \beta^{-1}$ for $\beta \ll 1$ is not surprising; essentially it shows that the coupling matrix with $\beta = 0$ still induces synchronization.

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{pmatrix} \beta x + z^2 \\ xy + \beta \sin y \\ \beta z(1-x) \end{pmatrix} \quad \boldsymbol{\Gamma} = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \qquad \boldsymbol{H}(\boldsymbol{x}) = \begin{pmatrix} \beta x + y \\ \beta \sin y + z \\ \beta z(1-x) \end{pmatrix} \quad \boldsymbol{\Gamma} = \begin{pmatrix} \beta & 1 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{pmatrix}$$

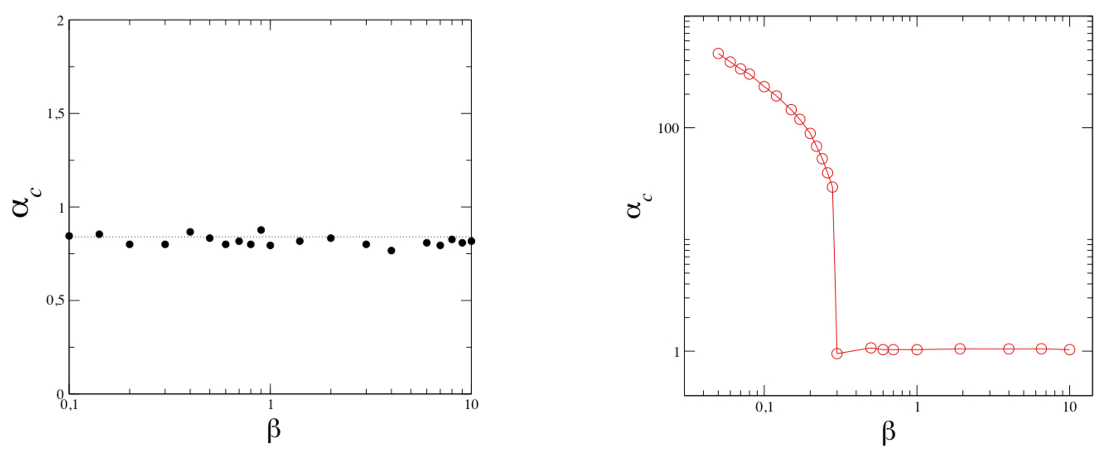


Figure 3. Simulation results for the critical α_c for the two coupling functions. For the first case, see left side, $\Gamma = \beta I$ is positive definite for $\beta > 0$, and the behavior of α_c does not depend significantly on β. For the second case, Γ is a Jordan block with eigenvalues equal to β. In this situation, for large values of β the critical coupling α_c appears independent of β, as opposed to the small values of β. In such case, the critical coupling scales as $\alpha_c \propto \beta^{-1}$.

4. Notation and Background

Here we quickly review results which are important to prove our Theorem.

The object of our study is a network of identical systems with interactions akin to diffusion. We consider m-dimensional vector spaces $\mathbb{R}^m$, and denote the elements $x \in \mathbb{R}^m$ as column vectors $x = (x_1, \cdots, x_m)^*$, where * stands for the transpose. We endow this space with the p-norms $\|x\|_p = (\sum_{i=1}^m |x_i|^p)^{1/p}$. For simplicity, unless otherwise stated, the norm $\|\cdot\|$ represents the L_∞−norm $\|x\| = \max_i |x_i|$. Once we introduce a norm on the vector space $\mathbb{R}^m$, we also view the space of linear operators on $\mathbb{R}^m$ as a normed space equipped with the induced norm. The choice of norm in our case is immaterial as all our vector spaces have finite dimension. Furthermore, given a complex number z, we denote its real part as $\mathrm{Re}(z)$.

Jaap: we should be careful since a change of norm adds to the overall constant; this can scale with n, m dimensions.

4.1. Stability of trivial solutions

Consider the linear differential equation

$$\frac{dv}{dt} = U(t)v \tag{7}$$

where $v \in \mathbb{R}^m$, for $m \geq 1$, and $U(t)$ is a bounded and continuous matrix function. Recall that solutions of Eq. (7) can be written in terms of the evolution operator

$$v(t) = T(t, s)v(s).$$

The point $x = 0$ is an equilibrium point. The time dependence in Eq. (7) introduces additional subtleties. Therefore, we want to have a stability condition that will also imply persistence under perturbations. Uniform asymptotic stability is such a condition; it is related to the evolution operator of the homogeneous equation in the following way.

Definition 3. *Let $T(t, s)$ be the evolution operator associated with Eq. (7). $T(t, s)$ is said to be a uniform contraction if*

$$\forall\, t \geq s \colon \|T(t, s)\| \leq K e^{-\eta(t-s)} \qquad \textit{with } \eta > 0. \tag{8}$$

The trivial solution of Eq. (7) is uniformly asymptotic stable if and only if the evolution operator is a uniform contraction. Moreover, uniform contractions have a rather important roughness property: they are not destroyed under small perturbations of the linear equations.

Proposition 4. *Suppose $U(t)$ is a continuous matrix function on $\mathbb{R}$ and consider Eq. (7). Assume that the fundamental matrix $T(t, s)$ satisfies the exponential estimate*

$$\forall\, t \geq s \colon \|T(t, s)\| \leq K e^{\rho(t-s)} \tag{9}$$

with $K > 0$ and $\rho \in \mathbb{R}$.

Consider a continuous matrix function $V(t)$ satisfying

$$\sup_{t \in \mathbb{R}} \|V(t)\| = \delta$$

then the evolution operator $\hat{T}(t,s)$ *of the perturbed equation*

$$\frac{dy}{dt} = [U(t) + V(t)]y,$$

satisfies the exponential estimate

$$\forall\, t \geq s \colon \|\hat{T}(t,s)\| \leq K e^{\hat{\rho}(t-s)},$$

where $\hat{\rho} = \rho + \delta K$*.*

Note that when $\rho = -\eta$ is negative, then $T(t,s)$ is a uniform contraction, and setting $\delta < \eta/K$ we find as a corollary that $\hat{\rho} < 0$, hence uniform contractions persist under small perturbations.

This is a standard persistence result, and the proof can be found, for example in Ref. [25, Prop. 1]. There are various criteria to obtain conditions for uniformly asymptotic stability. We shall use the following criterion for diagonal dominant matrices

Proposition 5. *Let* $U(t) = [U_{ij}(t)]$ *be a bounded, continuous matrix function on* $\mathbb{R}^m$ *and suppose there exists a constant* $\eta > 0$ *such that*

$$\mathrm{Re}(U_{ii}(t)) + \sum_{\substack{j=1,\\ j\neq i}}^{m} |U_{ij}(t)| \leq -\eta < 0, \tag{10}$$

for all $t \in \mathbb{R}$ *and* $i = 1, \cdots, m$*. Then the evolution operator* $T(t,s)$ *is a uniform contraction satisfying*

$$\forall\, t \geq s \colon \|T(t,s)\| \leq e^{-\eta(t-s)}.$$

The proof can be found in [25]. We use these fundamental results to prove our statements.

5. Synchronization: Stability analysis

Our goal is to prove that the synchronization manifold is locally attractive, and then to obtain bounds for the convergence of trajectories in an open neighborhood of the synchronization manifold towards the manifold.

To this end, we first obtain equations that govern the dynamics near the synchronization manifold. The dependence of f on time is not crucial in the proof, hence omitted for simplicity. Recall that we defined $\Gamma := DH(0)$. Using a tensor representation we can write the n equations as a single equation. First define

$$X = \mathrm{col}(x_1, \ldots, x_n),$$

where col denotes the vectorization formed by stacking the column vectors x_i into a single column vector. Similarly, we define

$$F(X) = \mathrm{col}(f(x_1), \cdots, f(x_n)).$$

We can analyze small perturbations away from the synchronization manifold in terms of the tensor representation

$$X = \mathbb{1} \otimes s + \xi, \tag{11}$$

where $\otimes$ is the tensor product, and $\mathbb{1} \in \mathbb{R}^n$ such that $\mathbb{1} = \mathrm{col}(1, \cdots, 1)$, which is the eigenvector of L corresponding to the eigenvalue zero. Note that $\mathbb{1} \otimes s$ defines the synchronization manifold, and we view ξ as a perturbation to the synchronized state.

Let us briefly expand a bit on the coordinate splitting in (11). Our state space $(\mathbb{R}^n \otimes \mathbb{R}^m, \|\cdot\|)$ can be canonically identified with $\mathbb{R}^{nm}$, which we will use for shorter notation. The norm need not come from an (Euclidean) inner product. For example, it can be the maximum over the Euclidean norm $\|\cdot\|_E$ on each of the $\mathbb{R}^m$ spaces, i.e.

$$\|(x_1, \dots, x_n)\| = \max_{1 \le i \le n} \|x_i\|_E \qquad \text{where } x_i \in \mathbb{R}^m. \tag{12}$$

The coordinate splitting (11) is associated to a splitting of $\mathbb{R}^{nm}$ as the direct sum of subspaces

$$\mathbb{R}^{nm} = M \oplus N \tag{13}$$

with associated projections

$$\pi_M \colon \mathbb{R}^{nm} \to M, \qquad \pi_N \colon \mathbb{R}^{nm} \to N.$$

The subspaces $M, N \subset \mathbb{R}^{nm}$ are determined by embeddings from $\mathbb{R}^m$ and $\mathbb{R}^{(n-1)m}$ respectively, induced by the Laplacian matrix L on $\mathbb{R}^n$.

Let us for the moment use the simplifying assumption that L is diagonalizable with eigenvectors $\mathbb{1}, v_2, \dots, v_n$. Then the subspaces M, N have natural representations in terms of these eigenvectors as

$$M = \mathrm{span}(\mathbb{1}) \otimes \mathbb{R}^m, \qquad N = \mathrm{span}(v_2, \dots, v_n) \otimes \mathbb{R}^m.$$

This means that we have 'natural' embeddings that induce coordinates on these subspaces:

$$\begin{aligned} \iota_M &\colon \mathbb{R}^m \to M, & s &\mapsto \mathbb{1} \otimes s = \mathrm{col}(s, \dots, s), \\ \iota_N &\colon \mathbb{R}^{(n-1)m} \to N, & (y_2, \dots, y_n) &\mapsto \sum_{j=2}^{n} v_j \otimes y_j. \end{aligned}$$

If we drop the assumption that L is diagonalizable then we lose the natural choice of an embedding for N.

The norm $\|\cdot\|$ on $\mathbb{R}^{nm}$ can be restricted to the subspaces M, N and induces norms on the 'coordinate' spaces $\mathbb{R}^m, \mathbb{R}^{(n-1)m}$ by pullback under the embeddings. That is, the induced norm on $\mathbb{R}^m$ is given for example by

$$\|s\|_{\mathbb{R}^m} = \|\iota_M(s)\| = \|\mathbb{1} \otimes s\|,$$

which is equal to $\|s\|_E$ if $\|\cdot\|$ is chosen to be the maximum of the Euclidean norms as in (12). Henceforth we shall identify $s \in \mathbb{R}^m$ with $\mathbb{1} \otimes s \in M$ under this isometry ι_M.

In the next proposition we represent the dynamics in terms of $s \in M$ and $\xi \in N$. The idea is that X is in an open neighborhood of M, i.e. that ξ is small.

Jaap: K conflicts with use as constant in exponential estimates

Proposition 6. *The perturbation ξ satisfies the equation*

$$\frac{d\xi}{dt} = K(s)\xi + R(s,\xi), \tag{14}$$

where

$$K(s) = I_n \otimes Df(s) - \alpha(L \otimes \Gamma)$$

and $R(s,\xi)$ is the remainder satisfying the following property: for any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\|\xi\| \leq \delta$ one has $\|R(s,\xi)\| \leq \varepsilon\|\xi\|$ uniformly in s, i.e. $R(s,\xi) \in o(\|\xi\|)$ uniformly in s.

Note that this equation for ξ still depends on s; later we shall provide stability bounds for it, independent of s.

Proof. Since H is smooth, we have by Taylor's theorem

$$H(x) = \Gamma\, x + r(x)$$

with $\|r(x)\| \leq \epsilon\|x\|$ for $\|x\| \leq \delta$. Now we define

$$\begin{aligned} R_H(X)_i &= \sum_{j=1}^n W_{ij} r(x_i - x_j) \\ &= \sum_{j=1}^n W_{ij} r(p_i(\mathbb{1} \otimes s + \xi) - p_j(\mathbb{1} \otimes s + \xi)) \\ &= \sum_{j=1}^n W_{ij} r(p_i(\xi) - p_j(\xi)), \end{aligned}$$

where p_i denotes projection onto the ith $\mathbb{R}^m$ tuple in $\mathbb{R}^{nm}$. So we see that $R_H(X) = R_H(\xi)$ does not depend on $s \in M$, and it satisfies the estimate

$$\|R_H(\xi)\| \leq \max_i \Big(\sum_{j=1}^n |W_{ij}| \Big) \epsilon\, 2\|\xi\| \qquad \text{when } \|\xi\| \leq \delta/2.$$

Recalling that $L_{ij} = \delta_{ij} S_i - W_{ij}$, the coupling term can then be rewritten as

$$\sum_{j=1}^n W_{ij} H(x_j - x_i) = -\sum_{j=1}^n L_{ij} \Gamma\, x_j + R_H(\xi)_i = (L \otimes \Gamma) X + R_H(\xi).$$

Let us now look at the term $F(X)$ describing the dynamics of the uncoupled nodes. We Taylor expand $X = \mathbb{1} \otimes s + \xi$ around $\mathbb{1} \otimes s$ and find

$$\begin{aligned} F(\mathbb{1} \otimes s + \xi) &= F(\mathbb{1} \otimes s) + DF(\mathbb{1} \otimes s)\xi + R_F(s,\xi) \\ &= \mathbb{1} \otimes f(s) + I_n \otimes Df(s)\xi + R_F(s,\xi), \end{aligned}$$

and we use a mean value theorem estimate to obtain that $\|R_F(s,\xi)\| \leq \epsilon\|\xi\|$ when $\|\xi\| \leq \delta$ and Df is uniformly continuous.

Thus we recover the full differential equation for the system, rewritten in the coordinates $(s, \xi) \in M \oplus N$ as

$$\frac{dX}{dt} = \mathbb{1} \otimes \frac{ds}{dt} + \frac{d\xi}{dt} = \mathbb{1} \otimes f(s) + I_n \otimes Df(s)\xi - \alpha(L \otimes \Gamma)\xi \\ + R_F(s, \xi) + \alpha R_H(\xi). \tag{15}$$

Next, we project both sides onto the spaces M, N to recover the separate differential equations for s, ξ respectively. This leads to

$$\dot{s} = f(s) + \pi_M(R_F(s, \xi) + \alpha R_H(\xi)), \tag{16}$$

$$\dot{\xi} = K(s)\xi + \pi_N(R_F(s, \xi) + \alpha R_H(\xi)), \tag{17}$$

where

$$K(s) = I_n \otimes Df(s) - \alpha(L \otimes \Gamma).$$

Note that both $I_n \otimes Df(s)$ and $L \otimes \Gamma$ preserve the subspaces M and N, since I_n and L preserve both $\mathrm{span}(\mathbb{1})$ and $\mathrm{span}(v_2, \ldots, v_n)$, thus the projections can be dropped there. Furthermore $L\,\mathbb{1} = 0$ means that the term $(L \otimes \Gamma)(\mathbb{1} \otimes s)$ disappears from the equation as well. □

Before we can continue to analyze the flow for ξ, we need to control s. Assumption *A1* that the single node system has a uniformly inflowing invariant set $U \subset \mathbb{R}^m$ leads to a similar result close to the synchronization manifold M in the coupled network. This guarantees that the solution for s stays inside $U \subset \mathbb{R}^m$, irrespectively of the precise details of its flow and the dependence of the flow on ξ.

To prove this result let us first introduce some additional notation. The following definition of uniform inflowing invariance was already referenced in assumption *A1*.

Definition 7 (ε-inflowing invariance)**.** *Let $\dot{x} = f(x)$ describe a dynamical system with $x \in \mathbb{R}^m$ and let $U \subset \mathbb{R}^m$ have C^1 boundary. Then we call U an ε-inflowing invariant set for f if we have at each point $x \in \partial U$ with inward-pointing normal vector n_x that*

$$\langle n_x, f(x) \rangle \geq \varepsilon. \tag{18}$$

Secondly, we need the concept of a tubular neighborhood of a submanifold.

Definition 8 (η-tubular neighborhood)**.** *Let $\mathbb{1} \otimes U \subset \mathbb{R}^{nm}$ be a subset of the synchronization manifold with C^1 boundary and let $\eta > 0$. Then we call*

$$U_\eta = \{\mathbb{1} \otimes s + \xi \mid s \in U,\, \xi \in N,\, \|\xi\| < \eta\} \tag{19}$$

an η-tubular neighborhood of $\mathbb{1} \otimes U$.

The following lemma implies that if a solution curve $(s(t), \xi(t))$ leaves U_η, then it must do so by $\|\xi(t)\|$ growing larger than η. As a result we can ignore the dependence on s of the flow for ξ as long as we have $\|\xi\| < \eta$: the coupling does not depend on s, and by assumption we have for $s \in U$ that the terms $Df(s)$ and $R_F(s, \xi)$ are uniformly bounded and small, respectively. This lemma is formulated with a general perturbation G to allow its reuse in the persistence proof.

Lemma 9. *Let the system $\dot{x} = f(t,x)$ satisfy assumption A1 with associated ε-inflowing invariant set $U \subset \mathbb{R}^m$. Let $\dot{X} = F(t,X)$ describe the dynamics of n uncoupled copies of this system and let $G\colon \mathbb{R} \times \mathbb{R}^{nm} \to \mathbb{R}^{nm}$ be a perturbation to F such that for some $r > 0$ and $\delta > 0$ it holds that*

$$\sup_{X \in U_r, t \in \mathbb{R}} \|G(t,X)\| \leq \delta < \varepsilon / \|\pi_M\|.$$

Then there exists an $0 < \eta \leq r$ such that solution curves $(s(t), \xi(t))$ of the system defined by $F + G$ can only leave the tubular neighborhood U_η through

$$\partial_{\mathrm{cyl}} U_\eta = \{\mathbb{1} \otimes s + \xi \mid s \in U,\ \|\xi\| = \eta\}.$$

Proof. Let us consider some $0 < \eta \leq r$ to be fixed later. Since U_η is cylinder-like, its boundary consists of two parts:

$$\partial U_\eta = \partial_{\mathrm{cyl}} U_\eta \cup \partial_{\mathrm{side}} U_\eta,$$

that is, the points $\mathbb{1} \otimes s + \xi$ where $\|\xi\| = \eta$ and those where $s \in \partial U$. We only need to consider the second case, that is $\partial_{\mathrm{side}} U_\eta$.

If n is an inward pointing normal vector at $s \in \partial U$, then $\mathbb{1} \otimes n$ points inwards to $\partial_{\mathrm{side}} U_\eta$ at $\mathbb{1} \otimes s + \xi$. Note that $\mathbb{1} \otimes n$ cannot be said to be normal to $\partial_{\mathrm{side}} U_\eta$ since a natural inner product on $\mathbb{R}^{nm}$ is missing. Instead we show that the component of $F+G$ projected along M has positive inner product with n, and thus $F+G$ points inwards at ∂U_η. Here we use the isometry ι_M to endow M with the inner product of $\mathbb{R}^m$.

From the bounds $\|DF\| = \|Df\| \leq B$ and $\|G\| \leq \delta$ it follows that

$$\begin{aligned}
&\langle n, \pi_M[F(\mathbb{1} \otimes s + \xi) + G(\mathbb{1} \otimes s + \xi)]\rangle \\
&= \langle n, f(s)\rangle + \langle n, \pi_M[DF(\mathbb{1} \otimes s + \tau\,\xi)\xi + G(\mathbb{1} \otimes s + \xi)]\rangle \\
&\geq \varepsilon - \|\pi_M\|(\|DF\|\|\xi\| + \|G\|) \\
&\geq \varepsilon - \|\pi_M\|(B\eta + \delta).
\end{aligned}$$

We applied the mean value theorem with $\tau \in (0,1)$ as interpolation variable. Since $\|\pi_M\|\delta < \varepsilon$ it follows that there exists an $\eta > 0$ such that $F + G$ points inwards everywhere at $\partial_{\mathrm{side}} U_\eta$. □

To apply Lemma 9 to our system, we note that the terms in (15) that do not arise from F are all linear or higher order in ξ, so if we denote these collectively by G. Then G can be made smaller than a given $\delta > 0$ by restricting to a sufficiently small tubular neighborhood U_r for some $r > 0$. Thus, if we ensure that $\|\xi(t)\| \leq \eta$ for all t, for a solution curve that starts inside U_η, then it stays inside U_η.

5.1. Diagonalizable Case

For clarity we first tackle the case where L and Γ are diagonalizable. We obtain the general case via perturbations and the roughness of dichotomies.

Lemma 10 (Diagonalizable case)**.** *Let the network satisfy the assumptions A1–A3. Moreover, assume that the network is connected and that the Laplacian L and linearized coupling Γ are diagonalizable:*

$$L = P\Lambda P^{-1} \quad \textit{and} \quad \Gamma = QGQ^{-1}$$

where $\Lambda = diag(\theta_1, \theta_2, \dots, \theta_n)$ and $G = diag(\beta_1, \dots, \beta_m)$ are the eigenvalue matrices of Λ and Γ respectively. Let s denote a solution of $\dot{s} = f(t, s)$ with $s(t_0) = s_0 \in U$ and let $K = K(s(t))$ denote the associated linear flow for $\xi \in N$ as defined in Prop. 6. Then there exists an α_c such that for all

$$\alpha > \frac{\alpha_c}{\gamma}$$

the linear evolution operator $\bar{T}(t, t_0) = T(t, t_0)|_N$ for ξ is a uniform contraction with estimate

$$\forall\, t \geq t_0\colon \|\bar{T}(t, t_0)\| \leq C\kappa(P) e^{-(\alpha\gamma - \alpha_c)(t - t_0)} \tag{20}$$

where $C = C(Q) > 1$.

Proof. Note that $S = P \otimes Q$ is an invertible matrix that diagonalizes $L \otimes \Gamma$ and the change of coordinates

$$\tilde{K} = S^{-1}\, K\, S = I_n \otimes Q^{-1}\, Df(t, s(t))\, Q - \alpha\, \Lambda \otimes G \tag{21}$$

reduces K to m-block diagonal form. Thus we have

$$\tilde{K} = \bigoplus_{i=1}^{n} \tilde{K}_i \quad \text{where} \quad \tilde{K}_i = Q^{-1}\, Df(t, s(t))\, Q - \alpha\, \theta_i\, G.$$

Since $\tilde{K}$ is block diagonal, it preserves the splitting $\mathbb{R}^{nm} = \bigoplus_{i=1}^{n} \mathbb{R}^m$, and hence its associated flow $\tilde{T}(t, t_0)$ will also be of the form

$$\tilde{T}(t, t_0) = \bigoplus_{i=1}^{n} \tilde{T}_i(t, t_0), \tag{22}$$

where each $\tilde{T}_i(t, t_0)$ is the flow of $\tilde{K}_i$. Note that the spaces $\mathbb{R}^m$ correspond to $v_i \otimes \mathbb{R}^m$ under the coordinate change S and in particular $M = S\, \mathbb{R}^m_1$ where $\mathbb{R}^m_1$ denotes the first copy of the n spaces $\mathbb{R}^m$. Thus, restricting K to N corresponds to restricting $\tilde{K}$ to the blocks $i \geq 2$.

Now for each block we obtain

$$\frac{dy_i}{dt} = \tilde{K}_i(s) y_i = [J - \alpha\, \theta_i\, G] y_i$$

where $J := Q^{-1}\, Df(t, s(t))\, Q$. By assumption *A1* we have that $s(t) \in U$ for all $t > t_0$, and also

$$\sup_{t \in \mathbb{R}, s \in U} \|Q^{-1} Df(t, s) Q\| \leq \kappa(Q) B.$$

Now, to apply Proposition 5 to this equation, we want

$$\mathrm{Re}[J_{kk} - \alpha\theta_i G_{kk}] + \sum_{\substack{j=1,\\ j\neq k}}^{m} |J_{kj}(t)| < 0$$

to hold. Since $\mathrm{Re}(J_{kk}) \leq |J_{kk}|$ we therefore find that

$$\alpha > \frac{\sum_{j=1}^{m} |J_{ij}|}{\mathrm{Re}(\theta_i G_{kk})}.$$

Noting that $\sum_{j=1}^{m} |J_{ij}| \leq \kappa(Q)B$ and $\gamma \leq \mathrm{Re}(\theta_i G_{kk})$ we obtain as sufficient condition that $\alpha > \alpha_c := (\kappa(Q)B)/\gamma$. Therefore, the evolution operator $\tilde{T}_i(t, t_0)$ satisfies

$$\|\tilde{T}_i(t, t_0)\| \leq Ce^{-(\alpha\gamma - \kappa(Q)B)(t-t_0)}$$

where $C = C(m)$ is a constant depending only on the dimension of the isolated nodes. Finally, using Eq. (22) and changing back to the original coordinates it follows that

$$\begin{aligned}
\|T(t,s)|_N\| &= \left\| S\left(\bigoplus_{i\geq 2} \tilde{T}_i(t,t_0) \right) S^{-1}|_N \right\| \\
&\leq \kappa(S) \max_{i\geq 2} \left\| \tilde{T}_i(t,t_0) \right\| \\
&\leq \kappa(P)\, \kappa(Q)\, C(m)\, e^{-(\alpha\gamma - \kappa(Q)B)(t-t_0)}.
\end{aligned} \tag{23}$$

Note that S^{-1} maps M and N onto the first and last $n-1$ of the m-tuples in $\mathbb{R}^{nm}$ respectively, hence the restriction to N reduces to a direct sum over $i \geq 2$ after conjugation with S. Also, we are only considering the parts of S and S^{-1} restricted to $S^{-1}N$ and N respectively, but we can simply estimate $\kappa(S|_{S^{-1}N}) \leq \kappa(S)$. □

Jaap: prove statements of Theorem 1 now: that U is uniformly attracting (including nonlinearities) and that the exponential rate is $\alpha_c - \alpha\gamma < 0$.

6. Proof: A result on Persistence

Recall that we consider a network of almost-identical nodes described by Eq. (2):

$$\dot{x}_i = f_i(t, x_i) + \alpha \sum_{j=1}^{n} W_{ij} H(x_j - x_i),$$

where $f_i(t, x_i) = f(t, x_i) + g_i(t, x_i)$. We shall now prove the following more precisely restated version of Theorem 2.

Theorem 2' (persistence)**.** *Consider a network of diffusively coupled equations and a choice of $\alpha > \alpha_c/\gamma$ as in Theorem 1. Let (2) describe a perturbation of the network such that*

$$\|g_i(t, x)\| \leq \varepsilon_g \quad \textit{for all } t \in \mathbb{R}\,,\ x \in \mathbb{R}^m \textit{ and } i \in \{1, \ldots, n\}\,. \tag{24}$$

Then there exist $C, \delta > 0$ and $\varepsilon_0 > 0$ such that if $\varepsilon_g < \varepsilon_0$ and $\|x_i(s) - x_j(s)\| \leq \delta$ for any $i, j \in \{1, \ldots, n\}$, then

$$\|x_i(t) - x_j(t)\| \leq Ce^{-(\alpha\gamma - \alpha_c)(t-s)}\|x_i(s) - x_j(s)\| + \frac{C\varepsilon_g}{\alpha_c - \alpha\gamma} \tag{25}$$

for all $t \geq s$.

Note that the proof does not specifically depend on the fact that the perturbations g_i to the nodes are decoupled; the function G below can depend in an arbitrary way on the total state X as long as it is uniformly small.

Proof. Denote by

$$G(t, X) = \mathrm{col}(g_1(t, x_1), \ldots, g_1(t, x_n))$$

the perturbation for the whole system. Note that $\|G\| \leq \varepsilon_g$ holds. When $\mathbb{R}^n$ is endowed with the maximum norm, we have the following equivalence of norms:

$$\|\pi_N\|^{-1}\|\xi\| \leq \sup_{1 \leq i,j \leq n} \|x_i - x_j\| \leq \|\xi\| \tag{26}$$

for any $X = \mathbb{1} \otimes s + \xi = (x_1, \ldots, x_n) \in \mathbb{R}^{nm}$. Thus, the estimates $\|x_i(s) - x_j(s)\| \leq \delta_0$ in Theorem 2 imply $\|\xi\| \leq \|\pi_N\|\delta_0$. Since the assumptions *A1*–*A3* are uniform in time, we can again suppress explicit time dependence and restrict to the case $s = 0$.

The addition of G modifies the differential equations (16) and (17) for s and ξ. Lemma 9 guarantees that there exists again an η-tubular neighborhood U_η such that solutions of the complete system for (s, ξ) cannot 'escape along s', when ε_g, η are sufficiently small.

The differential equation for ξ is given by

$$\dot{\xi} = K(s)\xi + R(s, \xi) + \pi_N(G(\mathbb{1} \otimes s + \xi)), \tag{27}$$

c.f. Proposition 6. We denote by $\varepsilon(\delta)$ the continuity modulus of R.

Variation of constants yields that solutions of (27) satisfy

$$\xi(t) = T(t, 0)\xi(0) + \int_0^t T(t, \tau)\Big[R(s(\tau), \xi(\tau)) + G(\mathbb{1} \otimes s + \xi)\Big]\,\mathrm{d}\tau$$

with $s(t)$ a solution of the modified equation for s and $T(t, \tau)$ the evolution operator associated to $K(s(t))$.

Let us now assume that $\|\xi(0)\| \leq \|\pi_N\|\delta_0$ and $\|\xi(t)\| \leq \delta_1 < \eta$ for all $t \geq 0$. From the proof of Theorem 1 it follows that $\|T(t, \tau)\| \leq Ce^{\rho(t-\tau)}$ with $\rho = \alpha_c - \alpha\gamma < 0$. Then we readily estimate

$$\begin{aligned} \|\xi(t)\| &\leq Ce^{\rho t}\|\pi_N\|\delta_0 + \int_0^t Ce^{\rho(t-\tau)}(\varepsilon(\delta_1)\|\xi(\tau)\| + \|\pi_N\|\varepsilon_g)\,\mathrm{d}\tau \\ &\leq Ce^{\rho t}\|\pi_N\|\delta_0 + \frac{C}{-\rho}(\varepsilon(\delta_1)\delta_1\|\pi_N\|\varepsilon_g). \end{aligned} \tag{28}$$

Jaap: do an explicit Gronwall-type estimate for (28) to recover (25).

$\square$

7. Normal hyperbolicity

In the previous section we proved persistence of synchronization under the condition

$$\alpha_c - \alpha\gamma < 0.$$

The left-hand side of this inequality is precisely the rate of exponential attraction towards the synchronization manifold. Indeed, this condition persists under small perturbations, so solutions of the perturbed system still converge to approximately the synchronization manifold. However, there need not exist an invariant manifold in the perturbed system that precisely attracts all nearly solutions. For this to be true, the original synchronization manifold must be a normally hyperbolic invariant manifold, see [27, 28] for fundamental definitions and results and e.g. [29] for normal hyperbolicity in synchronization of networks.

The additional condition for normal hyperbolicity to hold is that the flow contracts less along the synchronization manifold than that it contracts in the normal directions. The estimate α_c on the Jacobian of f was used previously to estimate the maximum expansion that the flow of f can detract from the contraction in the normal direction; similarly $-\alpha_c$ provides an estimate for the maximum contraction along the synchronization manifold. Thus the so-called spectral gap condition for r-normal hyperbolicity is satisfied when

$$\alpha_c - \alpha\gamma < -r\,\alpha_c \qquad \text{with } r \geq 1.$$

When the synchronization manifold satisfies this normal hyperbolicity condition, then the stronger result holds that it persists under arbitrary C^1 small perturbations. This means that not only solution curves converge to close to the original synchronization manifold, but that there exists a persistent C^r synchronization manifold

$$\tilde{M} = \{x_i = h_i(s), s \in \mathbb{R}^m, 1 \leq i \leq n\}$$

where the functions h_i are uniformly C^1 close to the identity function, i.e. $\tilde{M}$ is close to M. Solution curves of the perturbed system converge at an exponential rate ρ close to $\alpha_c - \alpha\gamma$ to this manifold $\tilde{M}$, and moreover a stronger 'shadowing' or 'isochrony' property holds that any solution curve $X(t)$ that converges to $\tilde{M}$, actually converges at exponential rate ρ to a unique solution curve $X_{\tilde{M}}(t)$ on $\tilde{M}$ in the sense that there exists a C such that for all $t \geq 0$

$$\|X(t) - X_{\tilde{M}}(t)\| \leq Ce^{\rho t}.$$

Sharper estimates can be obtained however, when it is known that the

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